IRRATIONAL NUMBERS ARISING FROM CERTAIN DIFFERENTIAL EQUATIONS

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Niven [3] gave a simple proof that π is irrational. Koksma [2] modified Niven's proof to show that e^r is irrational for every non-zero rational r. Dixon [1] made a similar modification to show that π is not algebraic of degree 2. In this note, we prove a general theorem which gives Niven's and Koksma's results as easy corollaries. A suitable modification in our proof also gives Dixon's result.

Theorem 1. *Let G be non-trivial solution of the equation*

$$L(u) = p_0 u^{(n)} + p_1 u^{(n-1)} + \cdots + p_n u = 0$$

where p_i are rational numbers and $p_n \neq 0$. If $b > 0$ is such that $G(x) \geq 0$ on $[0, b]$ and $G^{(i)}(0)$, $G^{(i)}(b)$ are rational for $0 \leq i \leq n-1$, then b is irrational.

Proof. Without any loss of generality, we may suppose that the p_i are integers. Suppose b is rational and set $b = p/q, (p, q) = 1, p, q \in \mathbb{Z}$. Set $f_m(x) = 1/m!(qx)^m(p - qx)^m$, where m is a natural number. It is easy to see that $f_m^{(k)}(0)$ are integers for $k \geq 0$ and since $f_m(x) = f_m(b - x)$, the same is true of $f_m^{(k)}(b)$. Now define the sequence $\{t_k\}$ recursively as follows:

$$\begin{gathered} t_0 = 1, \\ p_n t_1 - p_{n-1} t_0 = 0, \\ p_n t_2 - p_{n-1} t_1 + p_{n-2} t_0 = 0, \\ p_n t_{n-1} - p_{n-1} t_{n-2} + \cdots + (-1)^{n-1} p_1 t_0 = 0, \\ p_n t_{n+r} - p_{n-1} t_{n+r-1} + \cdots + (-1)^n p_0 t_r = 0 \quad \text{for } r \geq 0. \end{gathered}$$

Clearly, $p_n^k t_k$ is an integer for $k \geq 0$. Let

$$F_m(x) = \sum_{r=0}^{2m} t_r f_m^{(r)}(x).$$

If L^* is the adjoint of L, we have

$$\begin{aligned} L^*(F_m(x)) &= \sum_{k=0}^{n} (-1)^k p_{n-k} F_m^{(k)}(x) \\ &= \sum_{k=0}^{n} (-1)^k p_{n-k} \sum_{r=0}^{2m} t_r f_m^{(r+k)}(x) \\ &= \sum_{s=0}^{2m} f_m^{(s)}(x) \sum_{r+k=s} (-1)^k p_{n-k} t_r = p_n f_m(x). \end{aligned}$$

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Letting

$$\begin{aligned} P(u,v) &= u\left[p_{n-1}v - \frac{d}{dx}(p_{n-2}v) + \cdots + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(p_0 v)\right] + \\ &\quad \frac{du}{dx}\left[p_{n-2}v - \frac{d}{dx}(p_{n-3}v) + \cdots + (-1)^{n-2}\frac{d^{n-2}}{dx^{n-2}}(p_0 v)\right] + \\ &\quad + \cdots + \frac{d^{n-1}u}{dx^{n-1}}(p_0 v), \end{aligned}$$

we have by Lagrange's identity,

$$F_m(x)L(G) - G(x)L^*(F_m(x)) = \frac{d}{dx}P(G, F_m),$$

so that

$$-\int_0^b p_n f_m(x) G(x)\, dx = [P(G, F_m)]_0^b$$

since $L(G) = 0$. As $p_n^k t_k$ is an integer, it follows that $p_n^{2m} F_m^{(w)}(x)$ is an integer for $x = 0$ and $b, w \geq 0$. Thus, if A denotes the products of the denominators of $G^{(i)}(0)$ and $G^{(i)}(b)$, $0 \leq i \leq n-1$ (when expressed in lowest terms), $Ap_n^{2m}[P(G, F_m)]_0^b$ is an integer for every m. Now

$$Ap_n^{2m}[P(G, F_m)]_0^b = -Ap_n^{2m+1}\int_0^b f_m(x) G(x)\, dx.$$

If B and C are such that $|G(x)| \leq B, |qx(p - qx)| \leq C$ on $[0, b]$ we have

$$0 < Ap_n^{2m+1}\left|\int_0^b f_m(x) G(x)\, dx\right| < \frac{bBAp_n^{2m+1}C^{2m}}{m!}.$$

If m is sufficiently large, the right hand side is < 1, giving a contradiction. Hence b is irrational. □

Corollary 1. (1) π^2 *is irrational, (hence so also is* π*).* (2) $\log r$ *is irrational for every rational* $r > 0, r \neq 1$*.* (3) $e^r, \sin r, \cos r, \cosh r, \sinh r$ *are irrational for every non-zero rational* r*.*

Proof. If π^2 is irrational, consider $y'' + \pi^2 y = 0$ which has a solution $(1/\pi)\sin \pi x$. For $b = 1$, we get a contradiction. This proves (1). (2) and (3) are proved similarly, using the equation $y' - y = 0$ or $y'' \pm y = 0$. □

The following theorem is more arithmetical in nature.

Theorem 2. *Let* G *be a non-trivial solution of* $y^{(n)} + ty = 0$ *where* $t = (u/v,)(u, v) = 1$*, is a non-zero rational. Suppose* $G^{(i)}(0)$ *is rational for* $0 \leq i \leq n-1$ *and for some* $r \neq 0$ *with* $(r, n) = 1$*, we have* $G^{(r)}(0) \neq 0$*. If* β *is a non-zero rational, then* $G^{(n-1)}(\beta)$ *is irrational.*

Proof. Let $\beta = (a/b)$, $(a, b) = 1$. Define

$$f_p(x) = \frac{(\beta - x)^{np}[\beta^n - (\beta - x)^n]^{p-1} b^{np+(n-1)(p-1)}}{(p-1)!},$$

where p is a prime soon to be specified. If we compute the t_k in Theorem 1 for the equation $y^{(n)}+ty=0$, we find $t_k=0$ if $k\not\equiv 0 \pmod n$ and in case $k=sn$, $t_{sn}=(-1)^{sn-s}t^s$. If we set

$$F_p(x)=\sum_{k=0}^{M} t_k f_p^{(k)}(x)$$

were $M=n(2p-1)$, we have as in Theorem 1, $L^*(F_p(x))=tf_p(x)$, where L^* is the adjoint of $L(y)=y^{(n)}+ty$. Since f_p is a polynomial of degree M, $f_p^{(k)}(x)\equiv 0$ for $k>M$. If we set $\beta-x=y$ and $g_p(y)=y^{np}(\beta^n-y^n)^{p-1}$, then

$$g_p(y)=\sum_{i=0}^{p-1}(-1)^i\beta^{n(p-1-i)}\left(\begin{array}{c}p-1\\ i\end{array}\right)y^{n(p+1)}$$

from which it follows at once that $f_p^{(k)}(\beta)=0$ for all $k\neq n(p+i)$, $0\le i\le p-1$ and

$$f_p^{(k)}(\beta)=(-1)^{n(p+i)+i}\frac{b^{M-p+1}}{(p-1)!}\beta^{n(p-1-i)}\left(\begin{array}{c}p-1\\ i\end{array}\right)[n(p+i)]!$$

for $k=n(p+i)$. Hence $v^{2p-1}F_p(\beta)$ is an integer divisible by p. Since f_p has a zero of order $p-1$ at $x=0$, we have $f_p^{(k)}(0)=0$ for $k<p-1$. Writing $f_p(x)=(x^{p-1}/(p-1)!)h_p(x)$ we see from $(p-1)!f_p^{(k)}(x)=\sum_{s=0}^{k}\left(\begin{array}{c}k\\ s\end{array}\right)[x^{p-1}]^{(s)}[h_p(x)]^{(k-s)}$ that $f_p^{(k)}(0)=\left(\begin{array}{c}k\\ p-1\end{array}\right)h_p^{(k-p+1)}(0)$ for $k\ge p-1$. Clearly $f_p^{(k)}(0)$ is an integer as $h_p^{(k-p+1)}(0)$ is an integer. Also $\left(\begin{array}{c}k\\ p-1\end{array}\right)$ is divisible by p if $k\ge p$, and $k\not\equiv -1 \pmod p$. If $k\ge p$ and $k-p+1\equiv 0 \pmod p$, then $h_p^{(k-p+1)}(0)$ is divisible by p. If $k=p-1$, $f_p^{(p-1)}(0)=n^{p-1}a^{np+(n-1)(p-1)}$. Hence, $f_p^{(k)}(0)$ is divisible by p unless $k=p-1$. As $(r,n)=1$, let p be a prime $>na$, congruent to $-r \pmod n$. As $G^{(r)}(0)\neq 0$, and $p-1\equiv n-r-1 \pmod n$, the term $f_p^{(p-1)}(0)$ occurs once and only once in $\sum_{k=0}^{n-1}(-1)^kG^{(k)}(0)F_p^{(n-k-1)}$ and that is in the expression for $F_p^{(n-r-1)}(0)$. Let N be the product of all the denominators of the rationals $G(0),G'(0),\ldots,G^{(n-1)}(0),G^{(n-1)}(\beta)$. (Here we are supposing $G^{(n-1)}(\beta)$ is rational and will arrive at a contradiction). Thus, if $p>\max(na,NG^{(r)}(0),uv)$, all terms in

$$Nv^{2p-1}\{G^{(n-1)}(\beta)F_p(\beta)-\sum_{k=0}^{n-1}(-1)^kG^{(k)}(0)F_p^{(n-k-1)}(0)\}$$

are divisible by p except one term (the one involving $G^{(r)}(0)\neq 0$). Now, as in the proof of Theorem 1,

$$\begin{aligned}-uNv^{2p}\int_0^\beta G(x)f_p(x)\,dx = \quad & Nv^{2p-1}\{G^{(n-1)}(\beta)\\ & \times F_p(\beta)-\textstyle\sum_{k=0}^{n-1}(-1)^kG^{(k)}(0)F_p^{(n-k-1)}(0)\}.\end{aligned}$$

Thus, it follows $uNv^{2p}\int_0^\beta G(x)f_p(x)\,dx\neq 0$ for an infinity of primes p, using Dirichlet's theorem. On the other hand, we know $uNv^{2p}\int_0^\beta G(x)f_p(x)\,dx$ is an integer. This is a contradiction since

$$\lim_{p\to\infty}\left|uNv^{2p}\int_0^\beta G(x)f_p(x)\,dx\right| = 0.$$

This proves the theorem. □

Corollary 2. *Let p be an odd prime and G a non-trivial solution of $y^{(p)}+ty=0$, t a non-zero rational. If $G(0),\dots,G^{(p-1)}(0)$ are rational and at least two of them are non-zero, then $G(\beta),G'(\beta),\dots,G^{(p-1)}(\beta)$ are irrational for any non-zero rational β.*

Remark 1. The case $p=2$ has been covered by a corollary of Theorem 1.

Proof. As at least two of $G(0),G'(0),\dots,G^{(p-1)}(0)$ are non-zero, there is an r such that $G^{(r)}(0)\neq 0$ and $(r,p)=1$. The conditions of the theorem are satisfied and so $G^{(p-1)}(\beta)$ is irrational. As $G^{(i)}(x)$ also satisfies the conditions of the theorem for $0<i\le p-1$ the result follows. □

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References

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